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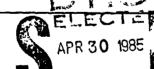16. DISTRIBUTION STATEMENT (of this Report)

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A magnetic body can be considered to consist of n sites, where n is large. The magnetic spins at these n sites, whose sum is the total magnetism present in the body, can be modelled by a triangular array of random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$. Standard theory of Physics would dictate that the joint

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Central Limit Theorems in the Area of Large

Deviations for Some Dependent Random Variables

by

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A magnetic body can be considered to consist of n sites, where n is large.
The magnetic spins at these n sites, whose sum is the total magnetism present in
the body, can be modelled by a triangular array of random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$.
Standard theory of Physics would dictate that the joint distribution of the spins
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central limit theorems for a wider class of Hamiltonians H_n, which are functions
of moment generating functions of suitable random variables. We also present
a number of examples to illustrate our theorems.

1. Introduction.

In this article we obtain central limit theorems for some dependent random variables which are used to describe the distribution of magnetic spins present in a ferromagnet crystal. A ferromagnet crystal consists of a large number of sites. The amount of magnetic spin present at site i will be denoted by $X_i^{(n)}$, i = 1, ..., n, where n is a positive integer. The magnetic spin present at any site interacts with the magnetic spins at its neighboring sites and hence gives rise to some dependency among the random variables $X_i^{(n)}$'s. In the Ising model, the joint distribution, at a fixed temperature T > 0, of the spin random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$, is given by

$$(1.1) \qquad dQ_n(\underline{x}) = z_n^{-1} \exp\left[-\frac{H_n(\underline{x})}{T}\right] \Pi dP(x_j),$$

where $\underline{x} = (x_1, \ldots, x_n) \in R^n$ and P is a probability measure on R. The function $H_n(\underline{x})$ is known as the Hamiltonian and it represents the energy of the crystal at the configuration $\underline{x}$, and z_n is a normalizing constant which is also known as the partition function. In many cases, an explicit evaluation of z_n is very difficult and physicists usually try to evaluate the limiting free spin per state $\xi(T)$, at the temperature T, defined as follows:

$$(1.2) \qquad \xi(T) = -\lim_{n \to \infty} [(\log(z_n))/n].$$

For some particular types of Hamiltonians, it has been shown by physicists that there exists a temperature level T_c such that the function $\xi(T)$ is infinite or finite according as $T < T_c$ or $T \geq T_c$ (see Kac (1968)). A phase transition is said to occur at the critical temperature T_c. As pointed out by Ellis and Newman (1978b), the existence of the critical temperature can be demonstrated in yet another way. We may be able to show that for $T > T_c$, there is a weak dependence among the random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$ and a standard central limit theorem is valid for $S_n/\sqrt{n}$ and that for $T = T_c$, there exists a $\delta \in (1,2)$ such that $S_n/n^{\delta/2}$ converges to a

non-Gaussian limit and for $T < T_c$, due to the strong dependence among the $X_i^{(n)}$'s, the random variables tend to cluster in several ergodic components. This shows a marked discontinuity in the asymptotic distribution of S_n as the temperature T is allowed to vary and represents our approach to demonstrating a phase transition.

In section 2, we consider a special case for the Hamiltonian, H_n, by setting it to be equal to $-\frac{1}{2n} \Sigma\Sigma x_i x_j$. This is known as the Curie-Weiss model. The asymptotic distribution of S_n for this model when P (which appears in Theorem 2.1) is symmetric Bernoulli is obtained by Simon and Griffiths (1973). In a two paper series, Ellis and Newman (1977, 1978a, 1978b) extended Theorem 2.1 of Simon and Griffiths to the class of probability measures L, defined in 2.2 (see also Ellis and Rosen (1979)). We state their extension precisely in Theorem 2.6. Recently Jong-Woo Jeon (1979) in his Ph.D. dissertation gave a simpler and statistically motivated proof of Theorem 2.6. The goal of this article is to extend Theorem 2.6 for a larger class of Hamiltonians H_n and probability measures P. Our main result, Theorem 3.7 is stated in Section 3. The proof of Theorem 3.7 relies on a recent large deviation local limit theorem of Chaganty and Sethuraman (1985), which is restated in Section 3 as Theorem 3.4 .

Let T_n, $n \geq 1$, be an arbitrary sequence of random variables with analytic moment generating function $\phi_n(z)$. We assume that T_n satisfies the conditions of Theorem 3.4 . In our generalized model the Hamiltonian $H_n(\underline{x})$ is taken to be equal to $- \log[\phi_n(s_n/n)]$, where $s_n = x_1 + \ldots + x_n$. Thus, the joint distribution of the spin random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$ is given by

$$(1.3) \qquad dQ_n(\underline{x}) = z_n^{-1} \phi_n(s_n/n) \, \Pi dP(x_j) ,$$

where P is an arbitrary probability measure. Let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$. Under appropriate conditions on the probability measure P we show in Theorems 3.7 and 3.18, there exists an integer $r \geq 1$ such that $S_n/n^{1-1/2r}$ converges in distribution to a random variable Y_r^*, which has a non-normal distribution when $r \geq 2$ and normal distribution when $r = 1$. The technique of our proof is to introduce a new random

variable W_n, conditional on which, $X_1^{(n)}, \ldots, X_n^{(n)}$ become i.i.d. It is easy to obtain the limiting distribution of W_n and the conditional asymptotic distribution of $S_n/n^{1-1/2r}$. Using the results of Sethuraman (1961) we deduce the asymptotic distribution of $S_n/n^{1-1/2r}$.

We now briefly give our reasons for calling these theorems on the asymptotic distribution of S_n under Q_n, defined in (1.3), as limit theorems in the area of large deviations. A standard technique to obtain the asymptotic distribution of S_n under Q_n is to first obtain the asymptotic distribution of S_n under P_n, where

$$(1.4) \qquad\qquad dP_n(\underline{x}) = \Pi dP(x_j)$$

and then to use contiguity arguments, as in LeCam (1960). This technique breaks down completely if $r \geq 2$. For the various models considered in Physics which are described in greater detail in Sections 2 and 3,

$$(1.5) \qquad |L_n(\underline{x})| = \left|\log \frac{dQ_n(\underline{x})}{dP_n(\underline{x})}\right| = \left|\frac{H_n(\underline{x})}{T} + \log z_n\right|$$

converges to ∞ in probability under P_n and thus contiguity arguments are not applicable here. Under P_n, $S_n/\sqrt{n}$ has a limiting normal distribution, also, $|L_n(\underline{x})|$ is small in the area of ordinary deviations of S_n, that is, when $S_n/\sqrt{n}$ is finite, while it is large otherwise. Thus from the point of view of P_n, we are looking for the asymptotic distribution of S_n, when P_n is modified by $L_n(\underline{x})$, which is substantially different from 1 in the area of large deviations of S_n. This view point helps in a statistically motivated proof of the asymptotic distribution of S_n under Q_n and describes the background behind the title of this article. One should also note that the normalizing factor on S_n in its asymptotic distribution under Q_n is different from the corresponding factor under P_n.

2. A Brief Summary of Curie-Weiss Model and Its Extensions.

In a ferromagnetic system with only isotropic pair interactions and with no external magnetic field, the Hamiltonian H_n, may be taken to be $-\frac{1}{2}\Sigma\Sigma a_{ij}x_i x_j$, where $a_{ij} \geq 0$. The Curie-Weiss model assumes that $a_{ij} = \frac{1}{n}$ for all i and j, that is to say that each spin interacts equally with every other spin with strength $\frac{1}{n}$ and takes P to be symmetric Bernoulli, i.e., $P(\{-1\}) = P(\{1\}) = \frac{1}{2}$. Replacing P by $P_T(x) = P(x\sqrt{T})$, we get

$$(2.1) \qquad dQ_n(\underline{x}) = z_n^{-1} \exp[s_n^2/2n] \, \Pi dP(x_j),$$

where $s_n = x_1 + \ldots + x_n$. This model has the advantage that the limiting free spin per state can be solved exactly. The existence of the critical temperature and phase transition for this model was demonstrated by Kac (1968). The asymptotic distribution for the total magnetism, S_n, for this model was obtained by Simon and Griffiths (1973). This is contained in Theorem 2.1.

<u>Theorem 2.1</u> (Simon and Griffiths). Let $X_j^{(n)}$, j = 1, ..., n be a triangular array of random variables whose joint distribution is given by (2.1) and P be symmetric Bernoulli. Then $S_n/n^{3/4}$ converges in distribution to a random variable whose density function is proportional to $\exp(-y^4/12)$.

Theorem 2.1 was extended to the class of probability measures L, which is defined below, by Ellis and Newman (1978b).

<u>Definition 2.2.</u> Let L be the class of probability measures P on R such that

$$(2.2) \qquad \int_{-\infty}^{\infty} \exp(x^2/2) \; dP(x) < \infty.$$

Fix $P \in L$. It can be shown that condition (2.2) guarantees the existence of the moment generating function (m.g.f.), m(u), of P. Let h(u) = log m(u) be the cumulant generating function (c.g.f.) of P. The function $G(u) = u^2/2 - h(u)$ plays an important role in Theorem 2.6 below.

Definition 2.3. A real number m is said to be a global minimum for G if

$G(u) \geq G(m)$ for all u.

Definition 2.4. A global minimum m for G is said to be of type r if

(2.3) $G(u+m) - G(m) = c_{2r} u^{2r}/(2r)! + O(|u|^{2r+1})$ as $u \to 0$,

where $c_{2r} = G^{(2r)}(m)$ is strictly positive.

Definition 2.5. A probability measure P is said to be pure if G has a unique global minimum.

Let Y_r, $r \geq 1$, be a sequence of random variables with density function $p_r(y)$, where

(2.4) $$p_r(y) = \begin{cases} d_r \ \exp \ [-c_{2r} y^{2r}/(2r)!\,] & \text{if } r \geq 2 \\ N(0, (1 - c_2)/c_2) & \text{if } r = 1 \end{cases}$$

and where d_r is the appropriate normalizing constant. With these definitions and notation we are now in a position to state the generalization of Theorem 2.1, due to Ellis and Newman (1978b).

Theorem 2.6. (Ellis and Newman). Let $P \in L$. Let P be pure, that is, let m be the unique global minimum of type r for G. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables with joint distribution given by (2.1). Let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$. Then

(2.5) $(S_n - nm)/n^{1-1/2r} \xrightarrow{d} Y_r$,

where Y_r is a random variable with density function given by (2.4).

It is easily verified that the symmetric Bernoulli measure is pure and belongs to the class L with the corresponding value of r equal to 2. Thus Theorem 2.6 contains Theorem 2.1.

Note that the moment generating function $M(z)$ of the standard normal is given by $\exp(z^2/2)$. Thus we can write (2.1) as

$$(2.6) \qquad dQ_n(\underline{x}) = z_n^{-1} [M(s_n/n)]^n \ \Pi dP(x_j).$$

One might ask the question whether it is possible to obtain limit theorems of the type (2.5) when $[M(z)]^n$ is replaced by the m.g.f. $\phi_n(z)$ of a random variable T_n, satisfying some conditions. We answer this question in the affirmative in the next section.

3. Further Extensions of the Curie-Weiss Model.

In this section we propose to extend Theorem 2.6 by enlarging the class of Hamiltonians as well as the class of probability measures L. The large deviation local limit theorems for arbitrary sequence T_n, $n \geq 1$, of random variables of Chaganty and Sethuraman (1985) (stated below) plays a key role in this extension. The Hamiltonians, H_n, in our generalized model (3.13) are taken to be the cumulant generating functions of these random variables T_n.

Let $\{T_n, n \geq 1\}$ be a sequence of non-lattice valued random variables with m.g.f.'s $\phi_n(z)$, $n \geq 1$, which are analytic and non-vanishing for z in $\Omega = \{z : |\text{Real}(z)| < c\}$, with $0 < c \leq \infty$. Let $I = (-a, a)$ and $\Omega_a = \{z : |\text{Real}(z)| < a\}$, where $0 < a < c$. Let

$$(3.1) \qquad \psi_n(z) = \frac{1}{n} \log \phi_n(z) \ , \quad \text{for } z \in \Omega$$

$$(3.2) \qquad \text{and} \qquad \gamma_n(u) = \sup_{|s| < c} [us - \psi_n(s)] \ , \quad \text{for } u \in R.$$

Let $A_n = \{\psi_n'(s); s \in I\}$. For $u \in A_n$, we have $\gamma_n(u) = [us_n - \psi_n(s_n)]$, where $s_n \in I$ satisfies $\psi_n'(s_n) = u$. Let P be a probability measure which satisfies the following condition:

$$(3.3) \qquad \int_{-c}^{c} \exp[\psi_n(x)] \ dP(x) < \infty \text{ for all } n \geq 1.$$

Let $h(u)$ denote the c.g.f. of P. It is easy to check that condition (3.3) implies that $h(u)$ is finite for $u \in R_n$, where

$$(3.4) \qquad R_n = \{u : \gamma_n(u) < \infty\}.$$

Let

$$(3.5) \qquad V_n(u) = \begin{cases} \gamma_n(u) - h(u) & \text{for } u \in R_n \\ \infty & \text{for } u \notin R_n. \end{cases}$$

The function V_n plays the same role as the function G of Section 2.

Definition 3.1. Let L* be the class of all probability measures P on $(-c, c)$ satisfying condition (3.3). We assume that there exists ℓ, $p_1 > 0$ such that

$$(3.6) \qquad \int_{R_n} \exp[-\ell V_n(u)] \; du = O(n^{p_1}),$$

and V_n's have a unique global minimum at some point m_n. Furthermore there exists $\eta_1 > 0$ such that

$$(3.7) \qquad \inf_{|u| > \delta} [V_n(m_n+u) - V_n(m_n)] = [V_n(m_n \pm \delta) - V_n(m_n)] \text{ for all } 0 < \delta < \eta_1.$$

Remark 3.2. Condition (3.7) is used mainly in inequality (3.27) of Lemma 3.13. An easily verifiable sufficient condition for (3.7) is

$$(3.8) \qquad V_n'(u) > 0 \text{ for } u > m_n \text{ and } V_n'(u) < 0 \text{ for } u < m_n.$$

In all the examples of section 4 we will be verifying (3.8) instead of (3.7).

Remark 3.3. Suppose that $R_n = (-\infty, \infty)$. If $\gamma_n(u)/|u|$ converges to ∞ as $|u| \to \infty$, then condition (3.3) implies (3.7) as seen below:

$$(3.9) \quad \exp[-V_n(u)] = \exp[-\gamma_n(u) + h(u)]$$

$$= \exp[-\gamma_n(u)][\int_{|x| < A} \exp[ux] \; dP(x) + \int_{|x| > A} \exp[ux] \; dP(x)]$$

$$\leq \exp[-\gamma_n(u) + uA] + \int_{|x| > A} \exp[\psi_n(x)] \; dP(x)$$

$$\leq \exp[-|u|(\frac{\gamma_n(u)}{|u|} - A)] + \int_{|x| > A} \exp[\psi_n(x)] \; dP(x).$$

The right hand side can be made close to zero first by choosing A and then letting $|u| \to \infty$. This shows that $V_n(u) \to \infty$ as $|u| \to \infty$. Since m_n is the unique global minimum of V_n, this also shows that condition (3.7) holds.

Let $m_n \in A_n$. Then there is a τ_n in I such that $\psi_n'(\tau_n) = m_n$. For $t \in I$, define

(3.10) $$G_n(t) = \psi_n(\tau_n) + it\,m_n - \psi_n(\tau_n + it).$$

The following theorem, which provides an asymptotic expansion for the density function k_n of T_n/n in terms of the large deviation rate γ_n, is due to Chaganty and Sethuraman (1985).

Theorem 3.4. Assume the following conditions for T_n:

(A) There exists $\beta > 0$ such that $|\psi_n(z)| < \beta$ for $z \in \Omega_a$ and $n \geq 1$.

(B) There exists $\alpha > 0$ such that $\psi_n''(\tau) \geq \alpha$ for $\tau \in I$ and $n \geq 1$.

(C) There exists $\eta > 0$ such that for any $0 < \delta < \eta$,
$$\inf_{|t| \geq \delta} \text{Real }(G_n(t)) = \min[\text{Real}(G_n(\delta)), \ \text{Real}(G_n(-\delta))], \text{ for } n \geq 1.$$

(D) There exists $p > 0$ such that
$$\sup_{\tau \in I} \int_{-\infty}^{\infty} |\phi_n(\tau + it)/\phi_n(\tau)|^{1/n} \, dt = O(n^p).$$

Then

(3.11) $$k_n(m_n) = \left[\frac{n}{2\pi \psi_n''(\tau_n)} \right]^{1/2} \exp(-n\gamma_n(m_n))[1 + O(\tfrac{1}{n})].$$

Remark 3.5. When T_n is the sum of n i.i.d. random variables, condition (C) is automatically satisfied and conditions (A), (B), (D) are easy to verify, since they do not depend on n.

Remark 3.6. Suppose that $m \in nA_n$. Then there exists $\xi_n \in I$ such that $\psi_n''(\xi_n) = m$, for $n \geq 1$. In this case one can verify that $\sqrt{\psi_n''(\xi_n)}/\sqrt{\psi_n''(\tau_n)} = [1 + O(|m_n - m|)]$ and thus we can rewrite (3.11) as

The cumulant generating function of P is given by

$$(4.6) \quad h(u) = \log(3/2) - 3\log|u| + |u| + \log[|u|(1 + e^{-2|u|}) - (1 - e^{-2|u|})], \quad u \in R.$$

Therefore $u \in R$,

$$(4.7) \quad V(u) = \gamma(u) - h(u)$$

$$= [-1 + \sqrt{1+u^2}] + \log|u| + \log[-1 + \sqrt{1+u^2}] - |u|$$

$$- \log[|u|(1 + e^{-2|u|}) - (1 - e^{-2|u|})] + \log(4/3).$$

Part (I). We just need to verify (4.1). Now

$$(4.8) \quad V'(u) = \sqrt{1 + (1/u^2)} + (2/u) - u/(u\coth(u) - 1)$$

$$> 1 + (2/u) - u/(u\coth(u) - 1)$$

$$= \frac{(u^2 + u)(\coth(u) - 1) + u\coth(u) - 2}{(u\coth(u) - 1)} > 0$$

for $u \geq 2$, since $u\coth(u) - 1 > 0$ and $\coth(u) > 1$. Again

$$(4.9) \quad V'(u) = \frac{[2 + \sqrt{1 + u^2}][u\coth(u) - 1] - u^2}{u[u\coth(u) - 1]} \quad \text{for } u > 0.$$

By differentiation one can verify that

$$(4.10) \quad u\coth(u) - 1 \geq \frac{u^2}{3} - \frac{u^4}{45} \quad \text{for } 0 < u < 7.5$$

and

$$(4.11) \quad \frac{u^2}{3} - \frac{u^4}{45} \geq \frac{u^2}{2 + \sqrt{1 + u^2}} \quad \text{for } 0 < u < 2.19.$$

These two inequalities show that $V'(u) > 0$ in the region $0 < u \leq 2$. Since

$V''(0) = \gamma''(0) - h''(0) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$, V has a unique global minimum of type 1.

written down. Part (I) will show that V has a unique global minimum of type r at $m_n \equiv 0$. Part (II) will verify the four conditions of Theorem 3.4 or Theorem 3.16 depending on whether T_n is non-lattice or lattice. Part (III) will verify conditions (3.3), (3.6) and (3.8) that insure that $P \in L^*$, in the case of continuous P and conditions (3.3), (3.8) and (3.38) that insure that $P \in L_1^*$, in the case of discrete P. These three parts will imply that the random variable S_n, after proper normalization converges in distribution to Y_r^*, for appropriate value of r.

The verifications of parts (II) and (III) are usually very easy. The verification of Part (I) presents more difficulties. However, in all the examples considered, V is symmetric around the origin and hence Part (I) is verified once we show that

(4.1) $V'(u) > 0$ for $u > 0$.

Note that (4.1) also verifies condition (3.8). The verification of (4.1) is routine in all the examples, but sometimes long, and therefore only the main steps are presented. One should also note that in part (II), condition (C) or (C') is automatically satisfied by Remark 3.5.

Example 4.1.   Let F have p.d.f. $(1/2) \exp(-|x|)$, $-\infty < x < \infty$, and P have p.d.f. $(3/4)(1 - x^2)$ for $|x| < 1$. Thus the joint distribution Q_n given in (3.13) becomes

(4.2) $dQ_n(\underline{x}) = z_n^{-1}(3/4)^n(1 - s_n^2/n^2)^{-n} \ \Pi(1 - x_j^2) \ \Pi dx_j.$

In this case

(4.3) $\phi_n(z) = (1 - z^2)^{-n}$ and

(4.4) $\psi(z) = -\log(1 - z^2)$ for $|Real(z)| < 1,$

(4.5) $\gamma(u) = \log 2 + [-1 + \sqrt{1 + u^2}\,] - 2 \log |u| + \log [-1 + \sqrt{1+u^2}\,]$, for $u \in R.$

where

(3.43) $g_n^*(y_n) = \dfrac{\sqrt{2\pi}\,\psi_n''(\xi_n)}{n^{(1-1/r)/2}}\; k_n(m_n + n^{-1/2r}y_n)\;\exp[n(h(m_n + n^{-1/2r}y_n) + V_n(m_n))].$

Imitating the proofs of Lemmas 3.9 thru 3.13, one can show the following:

(i) $\dfrac{1}{|h_n'|}\, g_n^*(y_n) \to g(y)$ as $n \to \infty$, uniformly on bounded intervals of y.

(ii) $\displaystyle\sum_{|y_n|\le n^{1/4r}} g_n^*(y_n) \to \int_{-\infty}^{\infty} g(y)\,dy$ as $n \to \infty$.

(iii) $\displaystyle\sum_{|y_n|> n^{1/4r}} g_n^*(y_n) \to 0$ as $n \to \infty$.

The above three steps (i), (ii), (iii) complete the proof of Lemma 3.19.

4. Applications.

In this section we illustrate the main theorems of section 3 with four applications and demonstrate limit theorems in quite complicated situations of dependent variables. The model (3.13) for the joint distribution of $(X_1^{(n)},\ldots,X_n^{(n)})$ is completely specified if T_n and P that arise in it are specified. To simplify matters, in all the examples of this section we let T_n be the sum of n i.i.d. random variables with common d.f. F. The four examples below contain all occurences of lattice and non-lattice T_n, and continuous and discrete P. The limit distribution of the normalized sum $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$ is normal ($r = 1$, in the notation of Theorems 3.7 and 3.18) in example 4.1 and is non-normal ($r=2$) in examples 4.2, 4.3 and 4.4. The results of Ellis and Newman (1978b) show that limit distributions with every possible value of $r > 2$ can also arise in suitable models.

In all the examples below we will specify F and P and write down the joint distribution Q_n. The exact expression for ϕ_n, $\psi_n \equiv \psi$, $\gamma_n \equiv \gamma$, h and $V_n \equiv V$ will be

If T_n satisfies the conditions of Theorem 3.16, then

$$(3.40) \qquad (S_n - n\tau_n)/n^{1-1/2r} \quad \xrightarrow{d} \quad Y_r^*,$$

where Y_r^* and τ_n are as defined in Theorem 3.7.

The proof of the above theorem parallels the proof of Theorem 3.7. We therefore outline briefly the modifications that need to be done. Note that dQ_n can be written as

$$(3.41) \qquad dQ_n(\underline{x}) = \sum_y \Pi dM_{n,y}(x_j) \ f_n^*(y),$$

where $f_n^*(y) = z_n^{-1} k_n(m_n + n^{-1/2r}y) \exp[n \ h(m_n + n^{-1/2r}y)]$ is a probability mass function of a lattice valued distribution with span $h_n' = h_n/n^{1-1/2r}$, and $dM_{n,y}(x_j)$ is as defined in (3.30). We introduce discrete random variables W_n^* with p.m.f. f_n^*. It suffices to show that W_n^* converges weakly to a continuous random variable W with probability density function f, defined in (3.35). The rest of the proof is identical to the proof of Theorem 3.7. Note that the span, h_n', of W_n^* converges to zero. By a theorem of Okamoto (1959), the sequence of random variables W_n^* will converge in distribution to W, once we prove the following:

Lemma 3.19. For $y \in R$, define $y_n = h_n'[y/h_n']$. Let the probability mass function f_n^* and the probability density function f be as defined above. Then

$$(3.42) \qquad \frac{1}{|h_n'|} \ f_n^*(y_n) \ \to \ f(y) \quad \text{as } n \to \infty,$$

uniformly on bounded intervals of y.

Proof (outline). Note that $f(y) = g(y)/\int g(y)dy$, where $g(y)$ is as defined in (3.15). We first write

$$f_n^*(y_n) = \frac{g_n^*(y_n)}{\Sigma g_n^*(y)} \ ,$$

(C') There exists $\eta > 0$ such that for any $0 < \delta < \eta$,

$$\inf_{\delta \le |t| \le \pi/|h_n|} \text{Real}(G_n(t)) = \min[\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))] \quad \text{for } n \ge 1,$$

where $G_n(t)$ is defined by (3.10).

(D') There exists $p > 0$ such that $|h_n|^{-1} = 0(n^p)$.

Then

$$(3.27) \qquad \frac{\sqrt{n}}{|h_n|} \ \Pr(\frac{T_n}{n} = m_n) = \left[\frac{1}{2\pi \ \psi_n''(\tau_n)}\right]^{1/2} \ \exp(-n\gamma_n(m_n))[1 + 0(\frac{1}{n})].$$

As before for a probability measure P on R, define $V_n(u)$ as in (3.5). The class of probability measures that are of interest is defined below.

Definition 3.17. Let L_1^* be the class of probability measures P satisfying conditions (3.3), (3.7) and (3.38)(defined below).

$$(3.38) \qquad \sum_{u \in R_n} \exp[-\ell V_n(u)] = 0(n^{p_1}) \text{ for some } \ell, \ p_1 > 0.$$

Note that (3.38) is the appropriate replacement of (3.6) for the lattice valued case.

For Hamiltonians which are functions of the moment generating functions of lattice valued random variables we have the following theorem almost identical to Theorem 3.7.

Theorem 3.18. Let $P \in L_1^*$. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables satisfying $|X_j^{(n)}| < c$ and having a joint distribution given by

$$(3.39) \qquad dQ_n(\underline{x}) = z_n^{-1} \ \phi_n(s_n/n) \ \Pi dP(x_j),$$

where ϕ_n is the m.g.f. of the lattice valued random variables T_n. Let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$. Let V_n have a unique global minimum of type r at the point $m_n \in A_n$. Let m_n converge to a point m belonging to the interior of $\cap A_n$.

The unconditional limiting distribution of $(S_n - n\tau_n)/n^{1-1/2r}$ is just the mixture

of the limiting conditional distribution and $f(y)$, by Theorem 3.15 of Sethuraman.

This completes the proof of Theorem 3.7. ||

Remark 3.14. When T_n is the sum of independent, normally distributed random

varibles with mean zero and variance one, $\phi_n(s_n/n)$ becomes $\exp[s_n^2/2n]$ and the

class of probability measures L^* reduces to the class L. Thus Theorem 3.7

generalizes Theorem 2.6 to a larger class of Hamiltonians and Probability measures.

We now state the theorem of Sethuraman (1961) which was crucially used to

obtain the limiting marginal distribution of $(S_n - n\tau_n)/n^{1-1/2r}$ in the proof of

Theorem 3.7.

Theorem 3.15 (Sethuraman). Let Λ_n be a sequence of probability measures on $V \times W$,

where V and W are topological spaces. Let μ_n be the marginal probability measure

of Λ_n on V and $\nu_n(v,\cdot)$ be the conditional probability measure on W. Suppose that

μ_n converges to a probability measure μ for every measurable set in V and for

almost all v with respect to μ, $\nu_n(v,\cdot)$ converges weakly to $\nu(v,\cdot)$. Then Λ_n

converges weakly to Λ, where

$$(3.36) \qquad\qquad \Lambda(A \times B) = \int_A \nu(v,B) \, d\mu(v)$$

for every measurable rectangular set $A \times B$.

We now turn our attention to the case where T_n, $n \geq 1$, are lattice valued

random variables with spans h_n, $n \geq 1$. The following theorem, which is analogous

to Theorem 3.4, was proved by Chaganty and Sethuraman (1985).

Theorem 3.16. Let T_n, $n \geq 1$, be a sequence of lattice valued random variables with

spans h_n, $n \geq 1$. Let m_n belong to the range of T_n/n. Assume that conditions (A),

(B) of Theorem 3.4 hold and replace conditions (C), (D) by the following:

We first note that

(3.32) $\log E_{M_{n,y}} \exp[t(S_n - n\tau_n)/n^{1-1/2r}]$

$$= n\left[-\frac{t\tau_n}{n^{1-1/2r}} + h\left(\frac{t}{n^{1-1/2r}} + m_{n,r}(y)\right) - h(m_{n,r}(y)) \right]$$

$$= n\left[-\frac{t\tau_n}{n^{1-1/2r}} + h'(m_{n,r}(y))\frac{t}{n^{1-1/2r}} + h''(m_{n,r}(y))\frac{t^2}{2n^{2-1/r}} + o(n^{-1}) \right]$$

$$= h''(m_n)ty + \frac{h''(m_n)t^2}{2\,n^{1-1/r}} + o(1) ,$$

since $\tau_n = h'(m_n)$.   Thus

(3.33) $\log E_{M_{n,y}} \exp[t(S_n - n\tau_n)/n^{1-1/2r}]$

$$\to \begin{cases} h''(m)ty & \text{if } r \geq 2 \\ h''(m)ty + \dfrac{h''(m)t^2}{2} & \text{if } r = 1. \end{cases}$$

This shows that the limiting distribution of $(S_n - n\tau_n)/n^{1-1/2r}$ given $W_n = y$ is degenerate at $h''(m)y$ if $r > 1$ and $N(h''(m)y, h''(m))$ if $r = 1$.   Next we note that

(3.34) $f_n(y) = z_n^{-1} n^{-1/2r} k_n(m_{n,r}(y)) \exp(n\, h(m_{n,r}(y)))$

$$= g_n(y) \Big/ \int g_n(y)\, dy ,$$

where $g_n(y)$ is as defined in (3.16).   By Lemmas 3.9, 3.11 and 3.13 it follows that

(3.35) $f_n(y) \to f(y) = \dfrac{g(y)}{\int g(y)\, dy} \quad \text{as } n \to \infty ,$

where $g(y) = \exp[-y^{2r} c_{2r}/(2r)!]$.   Thus the limiting distribution of W_n is $f(y)$.

Hence

$$\left| \int_{|y| > n^{1/4r}} g_n(y) \, dy \right| = O(n^q) \, \exp[-(n-\ell)[\frac{c_{2r,n}}{(2r)!} \frac{1}{\sqrt{n}} + K_n \, n^{-(2r+1)/4r}]]$$

which goes to zero since $|K_n| \le K$ for all n. The proof of Lemma 3.13 is now complete.

Proof of Theorem 3.7. We first express dQ_n defined in (3.13) as follows:

$$(3.28) \qquad dQ_n(\underline{x}) = z_n^{-1} \, \phi_n(s_n/n) \, \Pi dP(x_j)$$

$$= z_n^{-1} \int \exp(y \, s_n) \, k_n(y) \, dy \, \Pi dP(x_j).$$

Substituting $m_{n,r}(y) = m_n + n^{-1/2r}y$, we get

$$(3.29) \qquad dQ_n(\underline{x}) = z_n^{-1} \, n^{-1/2r} \int \exp(m_{n,r}(y) \, s_n) \, k_n(m_{n,r}(y)) \, dy \, \Pi dP(x_j)$$

$$= z_n^{-1} \, n^{-1/2r} \int \Pi \exp(x_j \, m_{n,r}(y) - h(m_{n,r}(y)) \, dP(x_j)$$

$$\cdot \, k_n(m_{n,r}(y)) \, \exp(nh(m_{n,r}(y))) \, dy$$

$$= \int \Pi dM_{n,y}(x_j) \, f_n(y) \, dy, \text{ where}$$

$$(3.30) \qquad dM_{n,y}(x_j) = \exp(x_j \, m_{n,r}(y) - h(m_{n,r}(y))) \, dP(x_j) \quad \text{and}$$

$$(3.31) \qquad f_n(y) = z_n^{-1} \, n^{-1/2r} \, k_n(m_{n,r}(y)) \, \exp(n \, h(m_{n,r}(y))).$$

Since $\int dQ_n(\underline{x}) = 1$ and $\int dM_{n,y}(x_j) = 1$ for each y and j, we have $\int f_n(y) dy = 1$. Thus we can introduce random variables W_n with probability density function $f_n(y)$ and the representation (3.29) of $dQ_n(\underline{x})$ shows that given $W_n = y$, $X_j^{(n)}$, $j = 1, \ldots, n$ are i.i.d. with common distribuiton $M_{n,y}(x)$.

We now proceed to obtain the limiting distribution of $(S_n - n\tau_n)/n^{1-1/2r}$ under $dM_{n,y}(x)$.

Substituting $z = n^{-1/2r}y$, we get

$$\left| \int_{|y|>n^{1/4r}} g_n(y) \; dy \right|$$

$$\leq \sqrt{2\pi\psi_n''(\xi_n)} \; n^{-(1-1/r)/2} \int_{|z|>n^{-1/4r}} \left| \exp[-n(V_n(m_n+z) - V_n(m_n))] \right|$$

$$\cdot \left| \exp(n \; \gamma_n(m_n + z)) \; k_n(m_n + z) \right| dz$$

$$\leq 0(n^{p+(1+1/r)/2}) \max_{|z|>n^{-1/4r}} \exp[-(n-\ell)(V_n(m_n + z) - V_n(m_n))]$$

$$\cdot \int \exp(-\ell(V_n(m_n + z) - V_n(m_n))) dz.$$

The last inequality follows from Lemma 3.12. This together with condition (3.6) yields

$$(3.27) \quad \left| \int_{|y|>n^{1/4r}} g_n(y) \; dy \right| \leq 0(n^q) \max_{|z|>n^{-1/4r}} \exp[-(n-\ell)(V_n(m_n + z) - V_n(m_n))]$$

$$= 0(n^q) \; \exp[-(n-\ell)L_n],$$

where

$$q = p_1 + p + (1 + 1/r)/2,$$

and

$$L_n = \min_{|z|>n^{-1/4r}}[V_n(m_n + z) - V_n(m_n)]. \quad \text{This minimum is attained at}$$

$z = \pm n^{-1/4r}$ by condition (3.7). Therefore,

$$L_n = \min[(V_n(m_n + n^{-1/4r}) - V_n(m_n)), \; (V_n(M_n - n^{-1/4r}) - V_n(m_n))]$$

$$= \frac{c_{2r,n}}{(2r)!} \; \frac{1}{\sqrt{n}} + K_n \; n^{-(2r + 1)/4r} \quad .$$

(3.24) $\sup_{y} |\exp(n\, \gamma_n(m_n + y))\, k_n(m_n + y)| = O(n^{p+1})$ as $n \to \infty$.

Proof. An application of the inversion formula yields (see (2.12) of Chaganty and Sethuraman (1985)),

$$|\exp[n((m_n + y)s - \psi_n(s))]\, k_n(m_n + y)|$$

$$= |\frac{n}{2\pi} \int_{-\infty}^{\infty} \exp[n(\psi_n(s + it) - \psi_n(s) - it(m_n + y))]\; dt|$$

$$\leq \frac{n}{2\pi} \int_{-\infty}^{\infty} |\phi_n(s + it) / \phi_n(s)|^{1/n} dt.$$

Taking supremum with respect to $s \in I$ and using condition (D) of Theorem 3.4 we get

$$\sup_{y} |\exp(n\gamma_n(m_n + y))\; k_n(m_n + y)| = O(n^{p+1}).\; \|$$

Lemma 3.13. Suppose that V_n has a unique global minimum at the point $m_n \in A_n$ and let g_n be as defined in (3.16). Then

(3.25) $\int_{|y| > n^{1/4r}} g_n(y)\; dy \to 0$ as $n \to \infty$.

Proof. Let $m_{n,r}(y) = m_n + n^{-1/2r}y$. By (3.16) we have

(3.26) $\int_{|y| > n^{1/4r}} g_n(y)dy = \left[\frac{2\pi\, \psi_n''(\xi_n)}{n}\right]^{1/2} \int_{|y| > n^{1/4r}} k_n(m_{n,r}(y))$

$$\cdot \exp[n(h(m_{n,r}(y)) + V_n(m_n))]\; dy$$

$$= \left[\frac{2\pi\, \psi_n''(\xi_n)}{n}\right]^{1/2} \int_{|y| > n^{1/4r}} k_n(m_{n,r}(y))$$

$$\cdot \exp\left[[-n(V_n(m_{n,r}(y)) - V_n(m_n))\,] + n\gamma_n(m_{n,r}(y))\right]\; dy.$$

Proof. Note that $n^{-1/2r}y$ converges to zero uniformly in y for $|y| < n^{1/4r}$.

Since m is an interior point of $\cap A_n$ there exists N_3 (independent of y) such that $m_{n,r}(y) = (m_n + n^{-1/2r}y) \in A_n$ for $n \geq N_3$. Applying Theorem 3.4 for $n \geq N_3$, we get

$$(3.22) \quad \int_{|y|\leq n^{1/4r}} g_n(y) \, dy = \left[\frac{2\pi \, \psi_n''(\xi_n)}{n}\right]^{1/2} \int_{|y|\leq n^{1/4r}} \exp[n(h(m_{n,r}(y)) + V_n(m_n))]$$

$$\cdot k_n(m_{n,r}(y)) \, dy$$

$$= \int_{|y|\leq n^{1/4r}} \exp[-n(V_n(m_{n,r}(y)) - V_n(m_n))]$$

$$\cdot [1 + O(|m_{n,r}(y) - m|) + O(\tfrac{1}{n})] dy$$

$$= \int_{-\infty}^{\infty} \lambda_n(y) \, dy \, ,$$

where

$$\lambda_n(y) = I(|y| \leq n^{1/4r}) \exp[-n(V_n(m_{n,r}(y)) - V_n(m_n))]$$

$$\cdot [1 + O(|m_{n,r}(y)-m|) + O(\tfrac{1}{n})] \, ,$$

and $I(\cdot)$ is the indicator function. It follows from Lemma 3.10 that $|\lambda_n(y)|$ is bounded by an integrable function. We can now conclude from Lemma 3.9 and Lebesgue dominated convergence theorem that

$$(3.23) \qquad \int_{-\infty}^{\infty} \lambda_n(y) \, dy \; \to \; \int_{-\infty}^{\infty} g(y) \, dy \quad \text{as} \quad n \to \infty.$$

The proof Lemma 3.11 is now complete.||

The next Lemma 3.12 is needed in the proof of Lemma 3.13.

Lemma 3.12. Let T_n, $n \geq 1$ be a sequence of random variables satisfying the conditions of Theorem 3.4. Then

(3.20) $$n[V_n(m_n + n^{-1/2r}y) - V_n(m_n)] \geq y^{2r} c_{2r}/2(2r)!$$

for all $n \geq N$, and $|y| < n^{1/4r}$.

__Proof.__ Let $0 < \epsilon < c_{2r}/2$. Since $c_{2r,n}$ converges to c_{2r} we can find N_1 such that $c_{2r,n} > c_{2r}/2 + \epsilon$ for all $n \geq N_1$. Recall that $\gamma_n(m_n) = m_n \tau_n - \psi_n(\tau_n)$, where τ_n is such that $\psi_n'(\tau_n) = m_n$. It is easy to verify that $\gamma_n'(m_n) = \tau_n$ and $\gamma_n''(m_n) = [\psi_n''(\tau_n)]^{-1}$. Also, $\gamma_n^{(2r+1)}(m_n)$ is the ratio of a simple function of $(2r+1)$ derivatives of ψ_n at τ_n to $[\psi_n''(\tau_n)]^{2r}$. Conditions (A) and (B) of Theorem 3.4 imply that all these derivatives are bounded uniformly in n and that $\psi_n''(\tau_n) \geq \alpha > 0$ (see (2.6) of Chaganty and Sethuraman (1985)). Hence $\gamma_n^{(2r+1)}(m_n)$ is uniformly bounded in n and consequently $V_n^{(2r+1)}(m_n) = \gamma_n^{(2r+1)}(m_n) - h^{(2r+1)}(m_n)$ is also uniformly bounded in n. Therefore

$$V_n(m_n+u) - V_n(m_n) = u^{2r} c_{2r,n}/(2r)! + K_n u^{2r+1}$$

as $u \to 0$, where $|K_n| \leq K < \infty$ for all n. Thus

$$n[V_n(m_n + n^{-1/2r}y) - V_n(m_n)]$$

$$= y^{2r} c_{2r,n}/(2r)! + K_n y^{2r+1}/n^{1/2r}$$

$$\geq y^{2r}c_{2r}/2(2r)! + y^{2r}[\epsilon/(2r)! - K y/n^{1/2r}]$$

$$\geq y^{2r}c_{2r}/2(2r)!$$

if $|y| < n^{1/4r}$ and $n \geq N = \max\{N_1, (K(2r)!/\epsilon)^{4r}\}$. This completes the proof of the Lemma 3.10. ||

__Lemma 3.11.__ Let g and g_n be as defined in (3.15) and (3.16). Then under the hypothesis of Lemma 3.9 we have

(3.21) $$\int_{|y| \leq n^{1/4r}} g_n(y) \, dy \to \int_{-\infty}^{\infty} g(y) \, dy \text{ as } n \to \infty.$$

For $y \in R$, let

$$(3.15) \qquad g(y) = \text{xp}[-y^{2r} c_{2r} / (2r)!] \quad \text{and}$$

$$(3.16) \quad g_n(y) = \left[\frac{2\pi \, \psi_n''(\xi_n)}{n} \right]^{1/2} k_n(m_n + n^{-1/2} r_y) \exp[n(h(m_n + n^{-1/2} r_y) + V_n(m_n))]$$

where ξ_n's are defined as in Remark 3.6. The functions g_n's arise in the proof of Theorem 3.7. Lemma 3.9 shows that $g_n(y)$ converges to $g(y)$ as $n \to \infty$ for each y. The next four lemmas, Lemma 3.10 - 3.13, show that

$$(3.17) \qquad \int_{-\infty}^{\infty} g_n(y) \, dy \;\to\; \int_{-\infty}^{\infty} g(y) \, dy \quad \text{as } n \to \infty.$$

Lemma 3.9. Suppose that V_n has a unique global minimum of type r at the point $m_n \in A_n$. Let m_n converge to m, where m is an interior point of $\cap A_n$. Suppose that $V_n^{(2r)}(m_n) = c_{2r,n}$ converges to c_{2r} as $n \to \infty$. Then

$$(3.18) \qquad g_n(y) \;\to\; g(y) \quad \text{as } n \to \infty.$$

Proof. Fix $y \in R$. Let $m_{n,r}(y) = m_n + n^{-1/2} r_y$. Then $m_{n,r}(y)$ converges to m and $m_{n,r}(y) \in A_n$ for sufficiently large n. Applying Theorem 3.4 together with Remark 3.6, with m_n replaced by $m_{n,r}(y)$ we get

$$(3.19) \qquad g_n(y) = \exp[-n\gamma_n(m_{n,r}(y)) + n(h(m_{n,r}(y)) + V_n(m_n))]$$

$$\cdot \left[1 + O(|m_{n,r}(y) - m|) + O(\tfrac{1}{n})\right]$$

$$= \exp[-n(V_n(m_{n,r}(y)) - V_n(m_n))] \left[1 + O(|m_{n,r}(y) - m|) + O(\tfrac{1}{n})\right]$$

$$= \exp[-y^{2r} c_{2r,n}/(2r)! + n \, o(|y|^{2r}/n)] \left[1 + O(|m_{n,r}(y) - m|) + O(\tfrac{1}{n})\right]$$

$$\to \; g(y) \quad \text{as } n \to \infty. \; \|$$

Lemma 3.10. Suppose that the V_n's have a unique global minimum of type r at the point $m_n \in A_n$. Then there exists N such that

$$(3.12) \qquad k_n(m_n) = \left[\frac{n}{2\pi \, \psi_n(\xi_n)} \right]^{1/2} e^{-n\gamma_n(m_n)} [1 + O(|m_n - m|) + O(\tfrac{1}{n})].$$

For each integer $r \geq 1$, let Y_r^* be a random variable with probability density function given by $d_r \exp[-c_{2r} \, y^{2r}/[h''(m)]^{2r} \, (2r)!]$ if $r \geq 2$ and $N(0, h''(m) [h''(m) + c_2]/c_2]$ if $r = 1$, where c_{2r} is the constant that appears in Theorem 3.7 below and d_r is the normalizing factor. With these assumptions and notation, we are in a position to state the main theorem of this section.

<u>Theorem 3.7.</u> Let $X_j^{(n)}$, $j = 1, \ldots, n$ be a triangular array of random variables satisfying $|X_j^{(n)}| < c$ and having a joint distribution given by

$$(3.13) \qquad dQ_n(\underline{x}) = z_n^{-1} \, \phi_n(s_n/n) \qquad \Pi dP(x_j),$$

where ϕ_n is the m.g.f. of T_n and $P \in L^*$. Assume that V_n, defined in (3.5), has a unique global minimum of type r at $m_n \in A_n$. Let $m_n \to m$ and $V_n^{(2r)}(m_n) \to c_{2r}$ as $n \to \infty$, where m is an interior point of $\cap A_n$. Let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$. If T_n satisfies the conditions of Theorem 3.4, then

$$(3.14) \qquad (S_n - n\tau_n) / n^{1-1/2r} \xrightarrow{d} Y_r^*,$$

where $\psi_n'(\tau_n) = m_n$ and Y_r^* is as defined above.

The proof of the above theorem is postponed until the end of Lemma 3.13.

<u>Remark 3.8.</u> The distribution function $Q_n(\underline{x})$ is well defined because

$$z_n = \int \exp[n \, \psi_n(s_n/n)] \quad \Pi dP(x_j)$$

$$\leq [\int_{-c}^{c} \exp[\psi_n(x)] \; dP(x)]^n < \infty,$$

wherein we have used condition (3.3) and the fact that ψ_n is a convex function.

Part (II). It is easy to verify that the random variables T_n, $n \geq 1$, satisfies

conditions (A), (B) of Theorem 3.4. Condition (D) also holds because

(4.12)
$$\sup_{|\tau| < a < 1} \int_{-\infty}^{\infty} |\phi(\tau + it)/\phi(\tau)| \, dt$$

$$= \sup_{|\tau| < a} \int_{0}^{\infty} \frac{2(1 - \tau^2)}{[4t^2 \tau^2 + (1 - \tau^2 + t^2)]^{\frac{1}{2}}} \, dt$$

$$\leq 2 + 2 \int_{1}^{\infty} \frac{1}{t^2} \, dt < \infty.$$

Part (III). To show $P \in L^*$, we only need to verify conditions (3.3) and (3.6).

Condition (3.3) trivially holds since $\exp[\psi(u)]$ is bounded on $(-1, 1)$, the support

of P. Condition (3.6) on the integrability follows from the fact $V(u) \sim \log |u|$

as $|u| \to \infty$.

We can therefore conclude in this example that

(4.13)
$$S_n / \sqrt{n} \xrightarrow{d} N(o, 1/3).$$

Example 4.2. Let F be the distribution function of the sum of two independent

and identically distributed uniform random variables on the interval $(-b, b)$ with

$b = \sqrt{3} / \sqrt{2}$. Let P be standard normal probability measure. The joint distribution

Q_n is given by

(4.14)
$$dQ_n(\underline{x}) = z_n^{-1} (2\pi)^{-n/2} \left[\frac{n \, \sinh(bs_n/n)}{(bs_n)} \right]^{2n} \exp\left[-\frac{1}{2} \Sigma x_j^2 \right] \Pi dx_j.$$

In this example

(4.15)
$$\phi_n(z) = [\sinh(bz)/(bz)]^{2n}$$

(4.16)
$$\psi(z) = 2[\log(\sinh(bz)) - \log(bz)] \quad \text{for } |\text{Real}(z)| < \infty .$$

Also

(4.17)
$$\gamma(u) = \sup_{|s| < \infty} [us - \psi(s)] \quad \text{for } |u| < 2b.$$

Thus

$$(4.18) \qquad V(u) = \gamma(u) - h(u) = \gamma(u) - (u^2/2)$$

$$= us_1 - \psi(s_1) - (u^2/2), \quad \text{for } |u| < 2b,$$

where $\psi'(s_1) = u$.

Part (I). Differentiating (4.18) with respect to u_1 we get

$$(4.19) \qquad V'(u) = [s_1 - \psi'(s_1)]$$

$$= [s_1^2 - 2bs_1 \coth(bs_1) + 2]/s_1, \quad \text{where } \psi'(s_1) = u.$$

Taking successive derivatives one can show that $t^2 - 3(t \coth(t) - 1) > 0$ for $t > 0$. Letting $t = bs_1$, and noting that $b = \sqrt{3}/\sqrt{2}$, we get

$$(4.20) \qquad s_1^2 - 2bs_1 \coth(bs_1) + 2 > 0 \quad \text{for } s_1 > 0.$$

Thus $V'(u) > 0$ for $u > 0$. With our choice of $b = \sqrt{3}/\sqrt{2}$ one can verify that $V''(0) = 0$ and $V^{(4)}(0) = 3/5 > 0$. Therefore V has a unique global minimum of order 2 at the origin.

Part (II). We need to verify that T_n satisfies conditions (A) thru (D) of Theorem 3.4. Note that $\psi''(0) = 1$. Hence, there exists $a > 0$ such that $\psi''(\tau) > \frac{1}{2} = \alpha$ for $|\tau| < a$. This verifies condition (B). It is easy to check that condition (A) also holds for this choice of a. Consider

$$\sup_{|\tau| < a} \left| \frac{\phi(\tau + it)}{\phi(\tau)} \right| = \left[\frac{b\tau}{\sinh(b\tau)} \right]^2 \left[\frac{(\sinh(b\tau))^2 + (\sin(bt))^2}{b^2(\tau^2 + t^2)} \right]$$

$$\leq \frac{c}{t^2}, \quad -\infty < t < \infty,$$

for some constant c. Therefore

(4.21) $\int\limits_{-\infty}^{\infty} |\phi(\tau + it)/\phi(\tau)| dt \leq \int\limits_{|t| \leq 1} 1\ dt + \int\limits_{|t| > 1} \frac{c}{t^2}\ dt < \infty$.

This verifies condition (D).

Part (III). We only need to verify that conditions (3.3) and (3.6). Let $\epsilon > 0$ be given. There exists a $\delta > 0$ such that $|x| < \delta$ implies $[\sinh(bx)/(bx)]^2 < (1+\epsilon)$. Therefore

$$(4.22) \quad \int\limits_{-\infty}^{\infty} \exp[\psi(x)]\ dP(x) = (2\pi)^{-1/2} \int\limits_{-\infty}^{\infty} [\sinh(bx)/(bx)] \exp(-x^2/2)\ dx$$

$$\leq (1 + \epsilon) + (2\pi)^{-1/2} \int\limits_{|x| > \delta} [\sinh(bx)/(bx)]^2 \exp(-x^2/2)\ dx$$

$$\leq (1 + \epsilon) + (b\delta)^{-2} (2\pi)^{-1/2} \int\limits_{|x| > \delta} [\sinh(bx)]^2 \exp(-x^2/2)\ dx,$$

which is finite. Hence condition (3.3) is satisfied. The other condition (3.6) also holds since the integral is over a finite range and $V(u) \geq -h(u)$.

Thus the random variables T_n's and P satisfy the conditions of Theorem 3.7. In this example

$$(4.23) \qquad S_n/n^{3/4} \overset{d}{\rightarrow} Y_2^* ,$$

where the p.d.f. of Y_2^* is given by $d_2 \exp(-y^4/40)$, $-\infty < y < \infty$.

Example 4.3. Let F be as defined in Example 4.2. Let P be symmetric Bernoulli, i.e., $P(\{-1\}) = P(\{1\}) = 1/2$. The joint distribution Q_n is given by

$$(4.24) \qquad dQ_n(\underline{x}) = z_n^{-1} \left[\frac{n\ \sinh(bs_n/n)}{(\sqrt{2}\ b\ s_n)} \right]^{2n} ,$$

where $x_j = \pm 1$ for all $1 \leq j \leq n$, and $b = \sqrt{3}\big/\sqrt{2}$. Recall that

(4.25) $\psi(z) = 2\ \log(\sinh(bz)) - 2\ \log(bz)$ for $|\text{Real}(z)| < \infty$,

and

(4.26) $\gamma(u) = us_1 - \psi(s_1)$ if $|u| < 2b$,

where $\psi'(s_1) = u$. The cumulant generating function of P is given by

$$(4.27) \qquad h(u) = \log \cosh(u), \quad -\infty < u < \infty .$$

The function $V(u) = \gamma(u) - h(u)$ is finite if $u < 2b$. Now for $u > 0$,

$$(4.28) \qquad V'(u) = s_1 - h'(\psi'(s_1))$$

$$= s_1 - \tanh(\psi'(s_1))$$

$$\geq s_1 - \psi'(s_1) > 0 ,$$

as shown in Example 4.2. Since the first three derivatives of V at the origin are equal to zero and $V^{(4)}(0) = 13/5$, the point zero is a minimum of order 2 for V. This completes the verification of Part (I). We have already checked Part (II) in Example 4.2. One can easily show that the probability measure P belongs to the class L^*, completing verification of Part (III). Thus by the conclusion of Theorem 3.7 we get

$$(4.29) \qquad S_n/n^{3/4} \xrightarrow{d} Y_2^* ,$$

where the p.d.f. of Y_2^* is given by $d_2 \exp[-13y^4/120]$, $-\infty < y < \infty$.

Example 4.4. Let F be symmetric Bernoulli distribution and P be the standard normal probability measure. The joint distribution in this example is given by

$$(4.30) \qquad dQ_n(\underline{x}) = z_n^{-1}(2\pi)^{-n/2} [\cosh(s_n/n)]^n \exp[-\tfrac{1}{2}\Sigma x_j^2] \, \Pi dx_j .$$

In this example

$$(4.31) \qquad \phi_n(z) = [\cosh(z)]^n ,$$

and

$$(4.32) \qquad \psi(z) = \log [\cosh(z)], \quad \text{for } |\text{Real}(z)| < \infty .$$

The large deviation rate $\gamma(u)$ is finite for $|u| < 1$ and is given by

(4.33)
$$\gamma(u) = us_2 - \log[\cosh(s_2)],$$

where $\tanh(s_2) = u$. Since the c.g.f. of P is $h(u) = u^2/2$, we get for $u > 0$,

(4.34)
$$V'(u) = s_2 - h'(\psi'(s_2))$$

$$= [s_2 - \tanh(s_2)] > 0$$

since $s_2 = \tanh^{-1}(u) > 0$. It is easy to check that the first three derivatives of V at the origin are zero and $V^{(4)}(0) = 2$. Thus zero is the unique global minimum of order 2. This completes the verification of Part (I). It is easy to verify that the lattice random variables T_n's satisfy all the conditions of Theorem 3.16.

Part (III). Condition (3.3) is trivially satisfied since $\cosh(x) \exp(-x^2/2)$ is an integrable function. Also for $\ell > 0$,

(4.35)
$$\sum_{u \in R_n} \exp[-\ell(V(u))] \leq \sum_{u \in R_n} \exp[\ell h(u)]$$

$$= \frac{n}{2} \sum_{u \in R_n} \frac{2}{n} \exp[\ell u^2/2]$$

$$\sim \frac{n}{2} \int_{-1}^{1} \exp[\ell u^2/2] = O(n).$$

Therefore P belongs to the class L_1^*. Thus by the conclusion of Theorem 3.18 we get

(4.36)
$$S_n / n^{3/4} \overset{d}{\to} Y_2^*,$$

where Y_2^* is distributed as $d_2 \exp(-y^4/12)$, $-\infty < y < \infty$.

REFERENCES

1) Chaganty, N.R. and Sethuraman, J. (1985). Large Deviation Local Limit Theorems for Arbitrary Sequences of Random Variables. To appear in Ann. of Probability.

2) Ellis, R.S. and Newman, C.M. (1978a). The Statistics of Curie-Weiss Models, J. Statist. Phys., 19, 149-161.

3) Ellis, R.S. and Newman, C.M. (1978b). Limit theorems for sums of dependent random variables occurring in Statistical Mechanics, Z. Wahrschein-lichkeitstheorie und Verw. Gebiete, 44, 117-139.

4) Ellis, R.S. and Rosen, J.S. (1979). Asymptotics of certain random fields on a circle. Random Fields, Edited by J. Fritz, J.L. Lebowitz and D. Saaz, Vol. I, 279-321. Seria Colloquia Mathematica Societatis Jańos Bolyai, No. 27.

5) Jong-Woo Jeon (1979). Central limit theorems in the regions of large deviations with applications to Statistical Mechanics, Ph.D. Dissertation, Department of Statistics, Florida State University, Tallahassee, FL.

6) Kac, M. (1968). Mathematical mechanisms of phase transitions, Statistical Physics: Phase Transitions and Superfluidity. Gordon and Breach, New York.

7) Le Cam, L. (1960). Locally asymptotic normal families of distributions. University of California Publ. Statist., 3, 37-98.

8) Okamoto, M. (1959). A convergence theorem for discrete probability distributions, Annals Inst. Stat. Math. 11, 107-112.

9) Sethuraman, J. (1961). Some limit theorems for joint distribution, Sankhya, Ser. A, 23, 379-386.

10) Simon, B. and Griffiths, R.B. (1973). The $(\psi^4)_2$ field theory as a statistical Ising model, Commun. Math. Phys., 33, 145-164.

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